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A Study of the Role of Modules in the failure of Systems

Abstract

Since the introduction of the concept of coherent systems and the description of the reliability of such systems in terms of the reliabilities of the components, the concept of *importance* of a component has created a new and fruitful area of research. Two distinct concepts of importance can be found in the literature. For a recent survey on this topic see Boland and El-Neweihi (1990). Birnbaum (1969), Natvig (1985), Boland, El-Neweihi and Proschan (1988), and others considered the improvement in the reliablity of the system which comes from the improvement of the reliablity of a component (which can be brought about by directly increasing the reliability of the component, or by augementing it in other ways) as the importance of that component. Fussell and Vesely (1972) and Barlow and Proschan (1975) on the other hand, defined the importance of a component to be the probability that the failure of the component caused the failure of the system. We take the view that the importance of a component or a module that is part of a system can be derived directly from the *role* of the component or the module in the failure of the system. Here again, it is possible that there will be several definitions of *role*. In this paper we will define the role of a module (or component) to be the probability that the module is among all the modules (or components) that failed at the time of system failure. With this definition of role, we can summarize the work of El-Neweihi, Proschan and Sethuraman (1978) as being mostly a study of the role of a cut set in a series-parallel system. We will refer to this paper in more detail later. The role of a module depends on the structure of the system in terms of the modules, the structure of the module in terms of its components and the distribution of lifetimes of the components. In this paper we study the role of a module under several structures and distributions for lifetimes. We establish various monotonicity properties and indicate applications of these properties to optimal allocation.

Another quantity that describes the nature of the components in sustaining the system is the number of components that fail at the time of the failure of the system. We establish monotonicity properties for the expected number of failed components and also indicate applications to optimal allocation.

A Study of the Role of Modules in the Failure of Systems

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Another quantity that describes the nature of the components in sustaining the system is the number of components that fail at the time of the failure of the system. We establish monotonicity properties for the expected number of failed components and also indicate applications to optimal allocation.

1. Introduction

Since the introduction of the concept of coherent systems and the description of the reliability of such systems in terms of the reliabilities of the components, the concept of *importance* of a component has created a new and fruitful area of research. Two distinct concepts of importance can be found in the literature. For a recent survey on this topic see Boland and El-Neweihi (1990). Birnbaum (1969), Natvig (1985), Boland, El-Neweihi and Proschan (1988), and others considered the improvement in the reliablity of the system which comes from the improvement of the reliablity of a component (which can be brought about by directly increasing the reliability of the component, or by augementing it in other ways) as the importance of that component. Fussell and Vesely (1972) and Barlow and Proschan (1975) on the other hand, defined the importance of a component to be the probability that the failure of the component caused the failure of the system. We take the view that the importance of a component or a module that is part of a system can be derived directly from the *role* of the component or the module in the failure of the system. Here again, it is possible that there will be several definitions of *role*. In this paper we will define the role of a module (or component) to be the probability that the module is among all the modules (or components) that failed at the time of system failure. With this definition of role, we can summarize the work of El-Neweihi, Proschan and Sethuraman (1978) as being mostly a study of the role of a cut set in a series-parallel system. We will refer to this paper in more detail later. The role of a module depends on the structure of the system in terms of the modules, the structure of the module in terms of its components and the distribution of lifetimes of the components. In this paper we study the role of a module under several structures and distributions for lifetimes. We establish various monotonicity properties and indicate applications of these properties to optimal allocation.

Another quantity that describes the nature of the components in sustaining the system is the number of components that fail at the time of the failure of the system. We establish monotonicity properties for the expected number of failed components and also indicate applications to optimal allocation.

To make our ideas more definite, consider a system S constructed from $k+1$ modules $P_0, P_1, \ldots, P_k$. We assume that P_i contains n_i components whose lifetimes have a common continuous distribution $F_i(x)$, $i = 0, \ldots, k$. We also assume that the $n_0 + \cdots + n_k$ components are independent. Let $\mathbf{n}$ denote $(n_1, \ldots, n_k)$. When $n_1 = \ldots = n_k = n$, we let n stand for $\mathbf{n}$. Similarly let $\mathbf{F}$ denote $(F_1, \ldots, F_k)$. When $F_1 = \ldots = F_k = F$, we let F stand for $\mathbf{F}$. We first consider the following structure $\mathbf{A}$ for S:

A1 The modules $P_0, P_1, \ldots, P_k$ are all parallel systems, and

A2 the system S is a $(k+1-r+1)$-out-of-$(k+1)$ system based on the $k+1$ modules $P_0, P_1, \ldots, P_k$.

This means that the system S fails as soon as r modules fail. Let T_i be the lifetime of the modules P_i, $i = 0, \ldots, k$ and let $R_0, R_1, \ldots, R_k$ be the ranks of $T_0, T_1, \ldots, T_k$. We will denote the probability that P_0 is among the r modules that failed first and caused the failure of the system by $P_r(n_0, F_0; \mathbf{n}, \mathbf{F}) = \text{Prob}\{R_0 \leq r\}$. A study of properties of the quantity $P_r(n_0, F_0; \mathbf{n}, \mathbf{F})$ is useful to determine the contribution of the module P_0 towards the failure of S. This quantity may be viewed as a measure of importance of the module P_0 in the spirit of the work of Barlow and Proschan (1975) and Fussell and Vesely (1972). In other parts of this paper we consider alternate structures to structure **A** and study properties of quantities analogous to $P_r(n_0, F_0; \mathbf{n}, \mathbf{F})$. The number of failed components, L, in all the modules at the time of the failure of S is another interesting piece of information on the system. We consider series-parallel systems S and study properties of L and its expectation later in the paper.

El-Neweihi, Proschan and Sethuraman (1978) considered a special case of the structure **A** above with $r = 1$ in **A2** and with $F_0 = \cdots = F_k = F$. They used a simple urn model to study properties of the probability that module P_0 causes the failure of S in terms of $\mathbf{n}$. Under some assumptions, they also proved the NBU property of the number of failed components L. Ross, Shahshahani and Weiss (1980) strengthened this result and proved that L has an IFR distribution.

Throughout this paper, we will use the concepts of majorization and arrangement increasing (AI) functions to establish various monotonicity properties. Marshall and Olkin (1979) is a good source for definitions and properties of these concepts.

This paper is organized as follows:

In Section 2 we derive a useful and compact expression for $P_r(n_0, F_0; \mathbf{n}, \mathbf{F})$. This allows us to obtain several qualitative properties for $P_r(n_0, F_0; \mathbf{n}, \mathbf{F})$. When $F_1 = \cdots = F_k = F$, we show that $P_r(n_0, F_0; \mathbf{n}, F)$ is Schur-concave in $\mathbf{n}$. These generalize the results of El-Neweihi, Proschan and Sethuraman (1978). When $n_1 = \cdots = n_k = n$ and $\bar{F}_i(x) = \exp(-\lambda_i R(x))$, $i = 1, \ldots, k$ (*the proportional hazards case*), we show that $P_1(n_0, F_0; n, \mathbf{F})$ is Schur-concave in $\boldsymbol{\lambda}$. Again we show that when $n_1 = \cdots = n_k = n$ and $F_i(x) = \exp(-\lambda_i A(x))$, $i = 1, \ldots, k$ (*the proportional left-hazards case*), we show that $P_r(n_0, F_0; n, \mathbf{F})$ is Schur-concave in $\boldsymbol{\lambda}$, for $r = 1, \ldots k + 1$. El-Neweihi (1980) showed that $P_1(n_0, F_0; \mathbf{n}, \mathbf{F})$ is an AI function in $\mathbf{n}$ and $\mathbf{F}$. By means of an example we show that this property does not hold, in general, for $P_r(n_0, F_0; \mathbf{n}, \mathbf{F})$ for $r \neq 1$. Applications of these results to optimal allocation models are illustrated.

In Section 3 we consider an alternate structure **B** for the system S. We replace **A1** and specialize **A2** as follows:

B1 The module P_i is a $a_i + 1 - \text{out} - \text{of}-n_i$ system, $i = 0, \ldots, k$, and

B2 the system is a series system based on $P_0, P_1, \ldots, P_k$.

The probability that the module P_0 causes the system to fail, $P_1(n_0, F_0; \mathbf{n}, \mathbf{F})$, will now be denoted by $P(a_0, n_0, F_0; \mathbf{a}, \mathbf{n}, \mathbf{F})$. We adopt the same conventions as before by writing a for $\mathbf{a}$, n for $\mathbf{n}$ and F for $\mathbf{F}$ when $a_1 = \cdots = a_k = a$, $n_1 = \cdots = n_k = n$ and $F_1 = \cdots = F_k = F$, respectively. When $a_1 = \cdots = a_k = a$, we show that $P(a_0, n_0, F_0; a, \mathbf{n}, \mathbf{F})$ is AI function in $\mathbf{n}, \mathbf{F}$. When $n_1 = \cdots = n_k = n$ and $F_1 = \cdots = F_k = F$, we show that $P(a_0, n_0, F_0; \mathbf{a}, n, F)$ is a Scur-concave function in $\mathbf{a}$. As before , applications of these results to optimal allocation models are illustrated.

Every coherent structure possesses a dual structure. The dual of a parallel structure is a series structure. The dual of a k-out-of-n structure is an $n - k + 1$-out-of-n structure. Thus we can consider duals of the structures **A** and **B** for the system S, based on modules $P_0, P_1, \ldots, P_k$, that have been studied in this paper. In other words assumptions **A1** can be replaced by

C1 The module P_i is a series system of its components, $i = 0, \ldots, k$.

Section 4 considers dual structures to those considered earlier and obtains analogous results.

In Section 5 we consider a series-parallel system S and study the number of failed components in all the modules at the time of the failure of the system. Since the module P_0 plays no special role here, we will consider only k modules $P_1, \ldots, P_k$ and denote the number of failed components by $L(\mathbf{n}, \mathbf{F})$. We derive the AI property of $E(\ L(\mathbf{n}, \mathbf{F})\ )$. For the parallel-series system where the components have exponential lifetimes, we prove that the expected number of failed components at the time of system failure is Schur-convex, when $k = 2$. We do not know if this property holds for the case $k \geq 3$.

2. The role of P_0 in the failure of system S

Consider a system S with structure **A** constructed from $k+1$ modules $P_0, P_1, \ldots, P_k$ as described in the previous section. The rank R_0 of T_0, the lifetime of P_0 among the lifetimes $T_1, \ldots, T_k$ of the modules $P_1, \ldots, P_k$, gives information on the role of P_0 in causing the failure of the system S. In particular, $P_r(n_0, F_0; \mathbf{n}, \mathbf{F}) = \text{Prob}\{R_0 \leq r\}$ is the probablity that P_0 is among the r modules that failed first and caused the failure of the system. Theroem 2.1 below gives a general expression for $P_r(n_0, F_0; \mathbf{n}, \mathbf{F})$. Let $h_{r|k}(p_1, \ldots, p_k) = P\{\sum_i^k Y_i \geq r\}$ where $Y_1, \ldots, Y_k$ are k independent Bernoulli random variables with parmeters $p_1, \ldots, p_k$. The quantity $h_{r|k}(p_1, \ldots, p_k)$ reperesents the reliability of an r-out-of-k system with k independent components having reliabilities $p_1, \ldots, p_k$.

Theorem 2.1

$$P_r(n_0, F_0; \mathbf{n}, \mathbf{F}) = 1 - \int h_{r|k}((F_1(x))^{n_1}, \ldots, (F_k(x))^{n_k}) dF_{T_0}(x) \qquad (2.1)$$

Proof: Notice that $P\{R_0 \geq r+1\} = P\{\sum_i^k Y_i \geq r\}$ where $Y_i = I\{T_i \leq T_0\}, i = 1, \ldots, k$. Conditional on T_0, the random variables $Y_1, \ldots, Y_k$ are k independent Bernoulli random variables with parameters $(F_1(T_0))^{n_1}, \ldots, (F_k(T_0))^{n_k}$. This immediately establishes (2.1). ◇

We will say that $F \leq G$ in the pointwise ordering if $F(x) \leq G(x)$ for all x. Notice that we use this pointwise ordering of distribution functions in this theorem and the rest of this paper, in contrast to the more popular stochastic ordering of distribution functions. Clearly $F \leq G$ if and only if $F \overset{st}{\geq} G$. The pointwise ordering allows us to state later results in standard notation (see Theorem 2.2, ...).

The next theorem gives some qualitative monotonicity properties of $P_r(n_0, F_0; \mathbf{n}, \mathbf{F})$.

Theorem 2.2

- **a** For each $F_0, \mathbf{n}, \mathbf{F}$, $P_r(n_0, F_0; \mathbf{n}, \mathbf{F})$ is non-increasing in n_0.
- **b** For each $n_0, \mathbf{n}, \mathbf{F}$, $P_r(n_0, F_0; \mathbf{n}, \mathbf{F})$ is non-decreasing in F_0 (with respect to the pointwise ordering of distribution functions).
- **c** For each $n_0, F_0, \mathbf{F}$, $P_r(n_0, F_0; \mathbf{n}, \mathbf{F})$ is non-decreasing in $\mathbf{n}$.
- **d** For each $n_0, F_0, \mathbf{n}$, $P_r(n_0, F_0; \mathbf{n}, \mathbf{F})$ is non-increasing in $\mathbf{F}$.

Proof: Notice that T_0 is stochastically increasing in n_0 and stochastically decreasing in F_0. Furthermore, $h_{r|k}((F_1(x))^{n_1}, \ldots, (F_k(x))^{n_k})$ is a non-decreasing function of x and $\mathbf{F}$, and non-increasing in $\mathbf{n}$. These facts establish **a**,bf b,**c** and **d**. ◇

We now assume that the lifetimes of all the components in modules $P_1, \ldots, P_k$ have the same distribution F. We will explore the monotonicity properties of $P_r(n_0, F_0; \mathbf{n}, F)$ in terms of $\mathbf{n}$ generalizing earlier results of El-Neweihi, Proschan and Sethuraman (1978).

Theorem 2.3 For each n_0, F_0 and F, $P_r(n_0, F_0; \mathbf{n}, F)$ is Schur-concave in $\mathbf{n}$.

Proof: In the study of order statistics from heterogeneous random variables Pledger and Proschan (1971) show in their Theorem 2.2 that $h_{r|k}((F(x))^{n_1}, \ldots, (F(x))^{n_k}) \geq h_{r|k}((F(x))^{n'_1}, \ldots, (F(x))^{n'_k})$ for each x, whenever $\mathbf{n} \overset{m}{\geq} \mathbf{n}'$. This theorem follows from this observation and Theorem 2.1. ◇

Remark 2.4. This theorem states that the module is more likely to be among the modules that fail before the failure of the system S when the sizes of the modules $P_1, \ldots, P_k$ are more homogeneous. This fact is intuitively more obvious when $r = 1$, the case considered in El-Neweihi, Proschan and Sethuraman (1978). It has been proved here for all values of r.

Remark 2.5. Let $P_{r*}(n_0, F_0; \mathbf{n}, F)$ be the probability that exactly r of the modules $P_0, P_1, \ldots, P_k$ have failed. We remark here that it is not true that $P_{r*}(n_0, F_0; \mathbf{n}, F)$ is Schur-concave in $\mathbf{n}$. For instance when $k = 2, r = 2$ and $F_0 = F_1 = F_2 = F$, we have $P_{r*}(n_0, F_0; \mathbf{n}, F) = \int_0^1 (x^{n_1} + x^{n_2} - 2x^{n_1+n_2}) n_0 x^{n_0 - 1} dx$, which is Schur-convex in $\mathbf{n}$, for each n_0. This remark shows that the claim in Theorem 3.8 in El-Neweihi (1980) is false.

We now assume that $n_1 = \cdots = n_k = n$ and that the life distribution F_i of the components of the module P_i have proportional hazards, *i.e.*, $\bar{F}_i(x) = \exp(-\lambda_i R(x)), i = 1, \ldots, k$. Then $P_r(n_0, F_0; n, \mathbf{F})$ is a function which depends on $\mathbf{F}$ only through $\boldsymbol{\lambda}$ and therefore may be denoted by $P_{r+}(n_0, F_0; n, \lambda)$. In Theorem 2.6 below we show that $P_{r+}(n_0, F_0; n, \lambda)$ is Schur-concave in $\boldsymbol{\lambda}$ when $r = 1$. We do not know whether this result will extend to other cases of r.

Theorem 2.6 $P_{1+}(n_0, F_0; n, \lambda)$ is Schur-concave in $\boldsymbol{\lambda}$.

Proof: Notice that $P_{1+}(n_0, F_0; n, \lambda) = \int \prod_{i=1}^k [1 - (1 - \exp(-\lambda_i x))^n] dF_{T_0}(x)$. It is easy to see that $f(\lambda)$ is log-concave by showing that the derivative of $\log f(\lambda)$ with respect to λ is decreasing. This proves that the integrand in $P_{1+}(n_0, F_0; n, \lambda)$ is Schur-concave in λ which implies that $P_{1+}(n_0, F_0; n, \lambda)$ is Schur-convave. $\diamond$

We now assume that $n_1 = \cdots = n_k = n$ and that the life distribution F_i of the components of the module P_i have proportional left-hazards, *i.e.*, $F_i(x) = \exp(-\lambda_i A(x)), i = 1, \ldots, k$. Then $P_r(n_0, F_0; n, \mathbf{F})$ is a function which depends on $\mathbf{F}$ only through $\boldsymbol{\lambda}$ and therefore may be denoted by $P_{r-}(n_0, F_0; n, \lambda)$. In Theorem 2.7 below we show that $P_{r-}(n_0, F_0; n, \lambda)$ is Schur-concave in $\boldsymbol{\lambda}$.

Theorem 2.7 $P_{r-}(n_0, F_0; n, \lambda)$ is Schur-concave in $\boldsymbol{\lambda}$.

Proof: Let $\boldsymbol{\lambda} \overset{m}{\geq} \boldsymbol{\lambda}'$. Then for each $x > 0$, $(nA(x))\boldsymbol{\lambda} \overset{m}{\geq} (nA(x))\ \boldsymbol{\lambda}'$. It follows that $h_{r|k}(\exp(-n\lambda_1 A(x)), \ldots, \exp(-n\lambda_k A(x))) \geq h_{r|k}(\exp(-n\lambda'_1 A(x)), \ldots, \exp(-n\lambda'_k A(x)))$ for each x by using Theorem 2.2 of Pledger and Proschan (1971). The result now follows from (2.1). $\diamond$

We now make some remarks on the joint monotonicity properties of $P_r(n_0, F_0; \mathbf{n}, \mathbf{F})$ in $\mathbf{n}, \mathbf{F}$. El-Neweihi (1980) considered the case $r = 1$ and showed that $P_1(n_0, F_0; \mathbf{n}, \mathbf{F})$ is an AI function of $(\mathbf{n}, \mathbf{F})$. The following example shows that this AI property is not generally true for other values of r.

Example 2.8 Let $k = 2$ and suppose that $n_1 \le n_2$ and $F_1 \le F_2$. Then $P\{R_0 = 3\} = \int (F_1(x))^{n_1}(F_2(x))^{n_2} dF_{T_0}(x)$, which is an AI function in $(\mathbf{n}, \mathbf{F})$. Thus $P_2(n_0, F_0; \mathbf{n}, \mathbf{F}) = 1 - P\{R_0 = 3\}$ is arrangement decreasing in $(\mathbf{n}, \mathbf{F})$.

Finally we end this section by illustrating applications of the results of this section to optimal allocation models. Without loss of generality suppose that $n_1 \ge \ldots \ge n_k$. Suppose that we have one more component that we can add to one of the modules $P_1, \ldots, P_k$. If we want to maximize, say, the expected value of R_0, we should add this component to P_1. This follows from Theorem 2.3 by observing that $(n_1 + 1, n_2, \ldots, n_k) \overset{m}{\ge} \cdots \overset{m}{\ge} (n_1, n_2, \ldots, n_k + 1)$.

The AI property of $P_1(n_0, F_0; \mathbf{n}, \mathbf{F})$ in $(\mathbf{n}, \mathbf{F})$ has the following interesting application. Consider a system S with structure $\mathbf{A}$ where $r = 1$. Suppose that the sizes $n_1, \ldots, n_k$ of the modules $P_1, \ldots, P_k$ are in increasing order. Suppose that we have collections of components with reliabilities $p_1 \ge \cdots \ge p_k$ at a particular time t. A careful examination of the proof of Theorem 4.8 in El-Neweihi (1980) shows that the reliability of S at time t is maximized by allocating components of reliability p_i to the module $P_i, i = 1, \ldots, k$.

3. The role of P_0 in an alternate structure for S

In this section we consider an alternate structure $\mathbf{B}$ for the system S. We replace **A1** and specialize **A2** as follows:

B1 The module P_i is an $a_i + 1$-out-of-n_i system, $i = 0, \ldots, k$, and

B2 the system S is a series system based on $P_0, P_1, \ldots, P_k$.

The probability that the module P_0 causes the system to fail, $P_1(n_0, F_0; \mathbf{n}, \mathbf{F})$, will now be denoted by $P(a_0, n_0, F_0; \mathbf{a}, \mathbf{n}, \mathbf{F})$. We adopt the same conventions as before by writing a for $\mathbf{a}$, n for $\mathbf{n}$ and F for $\mathbf{F}$ when $a_1 = \cdots = a_k = a$, $n_1 = \cdots = n_k = n$ and $F_1 = \cdots = F_k = F$, respectively. El-Neweihi, Proschan and Sethuraman (1978) considered the special case when $a_0 = \cdots = a_k = a$ and $F_0 = \cdots = F_k = F$ and showed that $P(a, n_0, F; a, \mathbf{n}, F)$ is Schur-concave in $\mathbf{n}$. In this section we study properties of $P(a_0, n_0, F_0; \mathbf{a}, \mathbf{n}, \mathbf{F})$ for more general situations. We will show in Theorem 3.2 that $P(a_0, n_0, F_0; a, \mathbf{n}, \mathbf{F})$ is AI function in $\mathbf{n}, \mathbf{F}$. To prove this we will need the following Lemma.

Lemma 3.1 Let $h_{(a+1)|n}(q)$ denote the reliability of an $a+1$-out-of-n system whose components are independent with identical reliability $p = 1 - q$. Then $h_{(a+1)|n}(q)$ is TP_2 in n, q, *i.e.*, $n_1 \le n_2$ and $q_1 \le q_2$ implies that

$$h_{(a+1)|n_1}(q_1) h_{(a+1)|n_2}(q_2) \ge h_{(a+1)|n_1}(q_2) h_{(a+1)|n_2}(q_1).$$

Proof: Notice that

$$h_{(a+1)|n}(q) = P\{B(n,p) \geq a+1\} = (a+1)\binom{n}{a+1} \int_q^1 t^{n-a-1}(1-t)^a dt,$$

where $B(n,p)$ is a binomial random variable with parameters n and p. To prove the lemma we need to show that $h_{(a+1)|n_2}(q)/h_{(a+1)|n_1}(q)$ is increasing in q, whenever $n_1 \leq n_2$. Differentiating this quotient with respect to q and neglecting constants and nonnegative terms we find that we have to show that

$$\int_q^1 \{t^{n_2}q^{n_1} - t^{n_1}q^{n_2}\}\frac{[(1-q)(1-t)]^a}{(qt)^{a+1}}dt \geq 0,$$

which follows from the fact that $t \geq q$ and $n_2 \leq n_1$ imply $\{t^{n_2}q^{n_1} - t^{n_1}q^{n_2}\} \geq 0$. ◇

Theorem 3.2 $P(a_0, n_0, F_0; a, \mathbf{n}, \mathbf{F})$ is AI in $\mathbf{n}, \mathbf{F}$, for each a_0, n_0, F_0, and a .

Proof: Let $n_1 \leq \cdots n_k$ and $\mathbf{F}, \mathbf{F}'$ be two vectors of distribution functions such that $F_i \leq F_j$ for some $i < j$ and $F_i' = F_j, F_j' = F_i$ and $F_l' = F_l$ for $l \neq i, l \neq j$. Then

$$\begin{aligned} &P(a_0, n_0, F_0; a, \mathbf{n}, \mathbf{F}) - P(a_0, n_0, F_0; a, \mathbf{n}, \mathbf{F}') \\ &= \int [h_{(a+1)|n_i}(F_i(x))h_{(a+1)|n_j}(F_j(x)) - h_{(a+1)|n_i}(F_j(x))h_{(a+1)|n_j}(F_i(x))] \\ &\quad \prod_{l\neq i, l\neq j} h_{(a+1)|n_l}(F_l(x))dG(x), \end{aligned}$$

where G is the distribution of the lifetime of the module P_0. Lemma 3.1 proved that the integrand in the integral above is nonnegative. This establishes the AI property of $P(a_0, n_0, F_0; a, \mathbf{n}, \mathbf{F})$. ◇

We now give an application of the above results to an optimal allocation problem. Let P_i be an $(a+1)$-out-of-n_i module, $i = 1, \ldots, k$ which are connected in series to form a system S. Suppose that the sizes $n_1, \ldots, n_k$ of the modules $P_1, \ldots, P_k$ are in increasing order. Suppose that we have collections of components with reliabilities $p_1 \geq \cdots \geq p_k$ at a particular time t. A careful examination of the proof of Theorem 3.2 above shows that the reliability of S at time t is maximized by allocating components of reliability p_i to the module $P_i, i = 1, \ldots, k$.

In Theorem 3.4 below we show that $P(a_0, n_0, F_0; \mathbf{a}, n, F)$ is Schur-concave in $\mathbf{a}$ to establish which we will prove the following lemma.

Lemma 3.3 Let X be a binomial random variable with parameters n, p . The distribution of X is IFR, which can be equivalently stated as $P\{X \geq k\}$ is log-concave in k, or $P\{X \geq k+1\}/P\{X \geq k\}$ decreases in k , or $P\{X = k\}/P\{X \geq k\}$ increases in k.

Proof: Notice that $P\{X = k\}/P\{X = k+1\} = (k+1)/(n-k)$ which increases in k. We will use this property to prove that $P\{X = k\}/P\{X \geq k\}$ increases in k, which will prove the theorem. Notice that

$$\begin{aligned}
&P\{X = k+1\}P\{X \geq k\} - P\{X = k\}P\{X \geq k+1\} \\
&= \sum_{m=k}^{n} P\{X = k+1\}P\{X = m\} - \sum_{m=k}^{n-1} P\{X = k\}P\{X = m+1\} \\
&= P\{X = k+1\}P\{X = n\} \\
&\quad + \sum_{m=k}^{n-1} [P\{X = k+1\}P\{X = m\} - P\{X = k\}P\{X = m+1\}] \\
&\geq 0.
\end{aligned}$$

$\diamondsuit$

Theorem 3.4 $P(a_0, n_0, F_0; \mathbf{a}, n, F)$ is Schur-concave in $\mathbf{a}$.

Proof: Notice that $P(a_0, n_0, F_0; \mathbf{a}, n, F) = \int (\prod_{i=1}^{k} P\{X \geq a_i + 1\}) dG(x)$, where X has a binomial distribution with parameters n, $\bar{F}(x)$ and G is the distribution of the lifetime of P_0. The integrand is Schur-concave in $\mathbf{a}$ because $P\{X \geq a+1\}$ is log-concave from Lemma 3.1. This establishes the theorem. $\diamondsuit$

4. Dual structures

Every coherent structure posesses a dual structure. The dual of a parallel structure is a series structure. The dual of a k-out-of-n structure is an $n-k+1$-out-of-n structure, and is a structure of the same type. Consider the system S with structure $\mathbf{A}$ based on the modules $P_0, P_1, \ldots, P_k$ as in Section 1. The dual of this is a system S' based on the modules $P_0', P_1', \ldots, P_k'$, consisting of $n_0, n_1, \ldots, n_k$ components, and possessing the structure $\mathbf{A}'$ as follows:

A′1 The modules $P_0', P_1', \ldots, P_k'$ are all series systems, and

A′2 the systems S' is an r-out-of-$k+1$ system based on the $k+1$ modules $P_0', P_1', \ldots, P_k'$.

This means that the system S' fails as soon as $k-r+1$ modules fail. Let T_i' be the lifetime of the modules P_i', $i = 0, \ldots, k$ and let $R_0', R_1', \ldots, R_k'$ be the ranks of $T_0', T_1', \ldots, T_k'$. Suppose that $T_i' = f(T_i)$ where f is a positive, strictly decreasing and continuous function. This happens, for instance when the lifetimes of the components in S' are the same function f of the lifetimes of the corresponding components of S. Let $P_r'(n_0, F_0'; \mathbf{n}, \mathbf{F}')$ be the probability that R_0' is less than or equal to r, that is P_0' is among the first r modules to fail in S'. We adopt the same conventions as before by writing n for $\mathbf{n}$ and F for $\mathbf{F}$ when $n_1 = \cdots = n_k = n$ and $F_1 = \cdots = F_k = F$, respectively.

It is easy to see that

$$P'_{k-r+1}(n_0, F'_0; \mathbf{n}, \mathbf{F}') = 1 - P_r(n_0, F_0; \mathbf{n}, \mathbf{F}), \tag{4.1}$$

that is, the probability that P'_0 is among the modules that caused the failure of the system S' is $1-$ the probability that P_0 is among the modules that caused the failure of the system S.

Theorems 4.2 to 4.5 below, stated without proofs, will illustrate how one can use equation (4.1) to establish properties for $P'_r(n_0, F'_0; \mathbf{n}, \mathbf{F}')$.

Remark 4.1 Note that if F and G are two possible distributions of a component in S such that $F \leq G$ and if F' and G' are the distributions of the corresponding component in S' then $F' \geq G'$. This fact will explain why the direction of some inequalities are unchanged when translating from S to S' in the theorems below.

Theorem 4.2

- **a** For each $F'_0, \mathbf{n}, \mathbf{F}'$, $P'_r(n_0, F'_0; \mathbf{n}, \mathbf{F}')$ is non-decreasing in n_0.
- **b** For each $n_0, \mathbf{n}, \mathbf{F}'$, $P'_r(n_0, F'_0; \mathbf{n}, \mathbf{F}')$ is non-decreasing in F'_0 (with respect to the pointwise ordering of distribution functions).
- **c** For each $n_0, F'_0, \mathbf{F}'$, $P'_r(n_0, F'_0; \mathbf{n}, \mathbf{F}')$ is non-increasing in $\mathbf{n}$.
- **d** For each $n_0, F'_0, \mathbf{n}$, $P'_r(n_0, F'_0; \mathbf{n}, \mathbf{F}')$ is non-increasing in $\mathbf{F}'$.

Theorem 4.3 For each n_0, F'_0, F', $P'_r(n_0, F'_0; \mathbf{n}, F')$ is Schur-convex in $\mathbf{n}$.

Theorem 4.4 The probability that P'_0 fails last among the $k+1$ modules, which is $1- P'_k(n_0, F'_0; \mathbf{n}, \mathbf{F}')$ is arrangement decreasing in $\mathbf{n}, \mathbf{F}'$.

Theorem 4.5 Let $\bar{F}'_i(x) = \exp(-\lambda_i R(x))$, $i = 1, \ldots, k$ (the proportional hazards case). Then $P'_r(n_0, F'_0; n, \mathbf{F}')$ is Schur-convex in $\boldsymbol{\lambda}$.

We will now consider the dual of the system S with the structure **B** defined in Section 1. This is a system S' modules $P'_0, P'_1, \ldots, P'_k$ satisfying the following structure.

B′1 The module P_i in an $(n_i - a_i)$-out-of-n_i system , $i = 0, \ldots, k$, and

B′2 the system S' is a parallel system based on the modules $P'_0, P'_1, \ldots, P'_k$.

We will denote the probability that P'_0 fails last by $P'(a_0, n_0, F'_0; \mathbf{a}, \mathbf{n}, \mathbf{F}')$. We adopt the same conventions as before by writing a for $\mathbf{a}$, n for $\mathbf{n}$ and F' for $\mathbf{F}$ when

$a_1 = \cdots = a_k = a$, $n_1 = \cdots = n_k = n$ and $F'_1 = \cdots = F'_k = F'$, respectively. Theorems 4.6 and 4.7 below, stated without proof, follow directly from equation (4.1) and Remark 4.1.

Theorem 4.6 For each a_0, n_0, F'_0 and a, $P'(a_0, n_0, F'_0; a, \mathbf{n}, \mathbf{F}')$ is arrangement decreasing in $\mathbf{n}, \mathbf{F}'$.

Theorem 4.7 For each a_0, n_0, F'_0 and F', $P'(a_0, n_0, F'_0; \mathbf{a}, n, F')$ is Schur-concave in $\mathbf{a}$.

5. Number of failed components at system failure

Consider a series-parallel system S based on k modules $P_1, \ldots, P_k$, which are parallel systems with sizes $n_1, \ldots, n_k$, respectively. Suppose that the common distribution of the lifetimes of components in P_i be F_i, $i = 1, \ldots, k$. Let $L(\mathbf{n}, \mathbf{F})$ be the number of failed components in all the modules at the time of failure of the system S. El-Neweihi, Proschan and Sethuraman (1978) derived interesting properties for $L(\mathbf{n}, F)$ when $F_1 = \cdots, F_k = F$. In particular they showed that $L(\mathbf{n}, F)$ has an NBU distribution. Ross, Shahshahani and Weiss (1980) improved upon this result by showing that $L(\mathbf{n}, F)$ has an IFR distribution. In this section we study properties of $E(L(\mathbf{n}, \mathbf{F}))$ where we do not assume that $F_1 = \cdots, F_k = F$.

Let $T_{ij}, j = 1, \ldots, n_i$ be the lifetimes of the n_i components in $P_i, i = 1, \ldots, k$. Let $T = \min_{1 \le i \le k} \max_{1 \le j \le n_i} T_{ij}$ be the lifetime of the sytem S. Clearly

$$L(\mathbf{n}, \mathbf{F}) = \sum_{i=1}^{k} \sum_{j=1}^{n_i} I\{T \ge T_{ij}\}, \tag{5.1}$$

where $I\{A\}$ is the indicator of the event A. The following lemma gives a useful expression for $E(L(\mathbf{n}, \mathbf{F}))$.

Lemma 5.1 $E(L(\mathbf{n}, \mathbf{F})) = \sum_{i=1}^{k} n_i \int \{\prod_{l=1, l \ne i}^{k} [1 - (F_l(x))^{n_l}]\} dF_i(x)$.

Proof: Let $T_i = \max_{1 \le j \le n_i} T_{ij}, i = 1, \ldots, k$. For each i, observe that $I\{T \ge T_{ij}\} = I\{\min(T_i, \min_{l \ne i} T_l) \ge T_{ij}\} = I\{\min_{l \ne i} T_l \ge T_{ij}\}, j = 1, \ldots, n_i$. From (5.1), we have

$$\begin{aligned} E(L(\mathbf{n}, \mathbf{F})) &= \sum_{i=1}^{k} E(\sum_{j=1}^{n_i} I\{T \ge T_{ij}\}) \\ &= \sum_{i=1}^{k} n_i E(I\{\min_{l \ne i} T_l \ge T_{i1}\}) \\ &= \sum_{i=1}^{k} n_i \int \{ \prod_{l=1, l \ne i}^{k} [1 - (F_l(x))^{n_l}]\} dF_i(x). \end{aligned}$$

◇

It is intuitively clear that $E(L(\mathbf{n},\mathbf{F}))$ is AI in $\mathbf{n},\mathbf{F}$. We prove this fact in Theorem 5.2 below.

Theorem 5.2 The expected number of failed components in the system S at the time of system failure $E(L(\mathbf{n},\mathbf{F}))$ is AI in $\mathbf{n},\mathbf{F}$.

Proof: Let $\mathbf{F},\mathbf{F}'$ be two vectors of distribution functions such that $F_i \leq F_j$ and $F_i = F_j', F_j = F'i, F_l = F_l'$ for $l \neq i, l \neq j$ and $n_1 \leq n_2 \leq \cdots \leq n_k$. Then

$$A(\mathbf{n},\mathbf{F}) - A(\mathbf{n},\mathbf{F}') = \sum_{r\neq i, r\neq j} n_r \int [\prod_{l\notin\{i,j,r\}} (1-(F_l(x))^{n_l})] \cdot$$
$$\cdot \{(1-(F_i(x))^{n_i})(1-(F_j(x))^{n_j}) - (1-(F_i(x))^{n_j})(1-(F_j(x))^{n_i})\} dF_r(x)$$
$$+ \int [\prod_{l\notin\{i,j\}} (1-(F_l(x))^{n_l})][n_j(1-(F_j(x))^{n_i}) - n_i(1-(F_i(x))^{n_j})] dF_j(x)$$
$$+ \int [\prod_{l\notin\{i,j\}} (1-(F_l(x))^{n_l})][n_j(1-(F_j(x))^{n_i}) - n_i(1-(F_i(x))^{n_j})] dF_i(x).$$

The function $(1-y^n)$ is TP_2 in y,n for $n = 0,1,\ldots, 0 < y < 1$. Hence the function $(1-y_1^{m_1})(1-y_2^{m_2})$ is AI in $(m_1,m_2),(y_1,y_2)$. This proves that the integrand in the first term for $A(\mathbf{n},\mathbf{F}) - A(\mathbf{n},\mathbf{F}')$ is nonnegative and hence it is nonnegative. Let $g(t) = n_j(1-t_i^n) - n_i(1-t_j^n)$. It is easy to see that $g(t)$ is a decreasing function of t for $0 \leq t \leq 1$. Let $c(x) = \prod_{l\notin\{i,j\}}(1-(F_l(x))^{n_l})$. The function $c(x)g(F(x))$ is decreasing in x. Using these facts and the inequalities $n_i \leq n_j, F_i \leq F_j$, we see that the sum of the last two terms for $A(\mathbf{n},\mathbf{F}) - A(\mathbf{n},\mathbf{F}')$ is equal to

$$\int c(x)g(F_i(x))dF_j(x) - \int c(x)g(F_j(x))dF_i(x)$$
$$\geq \int c(x)g(F_j(x))dF_j(x) - \int c(x)g(F_j(x))dF_i(x)$$
$$\geq 0.$$

This proves the theorem. ◇

Consider a parallel-series system S' with modules $P_1',\ldots,P_k'$ which are series systems with $n_1,\ldots,n_k$ components whose life distributions are $F_1',\ldots,F_k'$, respectively. Let $B(\mathbf{n},\mathbf{F}')$ be the expected number of failed components at the time of the failure of system S'. A consideration of the dual structure in Theorem 5.2 shows that $B(\mathbf{n},\mathbf{F}')$ is arrangement increasing in $(\mathbf{n},\mathbf{F}')$.

An implication of the above result to optimal allocation in a series-parallel system S is as follows. Let S be a series system consisting of modules $P_1,\ldots,P_k$ be k which are parallel systems with $n_1 \leq \cdots \leq n_k$ components, respectively. Suppose that we have collections of components with life distributions $F_1 \leq \cdots \leq F_k$. Then one should allocate components with life distributions $F_{(n-i+1)}$ to the module P_i to minimize the expected number of component failures at the time of the failure of system S.

We now consider a parallel-series system S' where the modules $P_1', \ldots, P_k'$ are series systems with the same number of components n. Assume further that $\bar{F}_i'(x) = \exp(-\lambda_i x), i = 1, \ldots, k$. We show in Theorem 5.3 below that, when $k = 2$, the expected number of component failures before system failure is Schur-convex in (λ_1, λ_2).

Theorem 5.3 Let $B(n, \mathbf{F}')$ be the expected number of component failures at system failure in the parallel-series system S' described above. Let $k = 2$. Then $B(n, \mathbf{F}')$ is Schur-convex in (λ_1, λ_2).

Proof: Let T_{ij}' be the lifetimes of the components of P_i' and let $T_i' = \min_{1 \leq j \leq n,} T_{ij}' i = 1, 2$. Then $T' = \max\{T_1', T_2'\}$ is the lifetime of the system S'. The number of component failures at system failure, $L'(n', \mathbf{F}')$ is given by $1 + \sum_{i=1}^{2} \sum_{j=1}^{n} I\{T' > T_{ij}'\}$. Since F_i' is exponential with parameter λ_i, it follows that

$$B(n, \mathbf{F}') = E(L'(n', \mathbf{F}')) = 1 + n\{\frac{\lambda_1}{\lambda_1 + n\lambda_2} + \frac{\lambda_2}{\lambda_2 + n\lambda_1}\}.$$

A direct calculation shows that $[\frac{\partial B(n,\mathbf{F}')}{\partial \lambda_1} - \frac{\partial B(n,\mathbf{F}')}{\partial \lambda_2}](\lambda_1 - \lambda_2) = d(\lambda_1 - \lambda_2)^2$, where $d \geq 0$ is some function of $\lambda_1 + \lambda_2$. This proves that $B(n, \mathbf{F}')$ is Schur-convex. ◊

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